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THERE IS NO BILINEAR BISHOP–PHELPS THEOREM*

BY

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ABSTRACT

We answer a question posed by R. Aron, C. Finet and E. Werner, on the bilinear version of the Bishop–Phelps theorem, by exhibiting an example of a Banach space X such that the set of norm-attaining bilinear forms on $X \times X$ is not dense in the space of all continuous bilinear forms.

Introduction

In their celebrated paper [B-P] E. Bishop and R. Phelps proved the by now classical result that the set of norm attaining linear functionals on a Banach space is dense in the dual space. They also asked if the result still holds for operators. More concretely, is the set $NA(X,Y)$ of norm-attaining operators between Banach spaces X and Y dense in the space $L(X,Y)$ of all bounded linear operators? A great deal of attention has been paid to this question over the last thirty years (see [L], [B], [G] for example).

In a forthcoming paper [A-F-W], R. Aron, C. Finet and E. Werner consider extensions of the Bishop–Phelps Theorem in a different, still very natural direction. Given a Banach space X, let us denote by $\mathcal{B}(X)$ the space of continuous bilinear forms on $X \times X$; let us say that $\varphi \in \mathcal{B}(X)$ attains its norm if there are $x_0, y_0 \in B_X$ (the unit ball of X) such that

$$|\varphi(x_0, y_0)| = \|\varphi\|,$$

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and let $\mathcal{B}_a(X)$ denote the set of norm-attaining bilinear forms.

The authors of [A-F-W] show that $\mathcal{B}_a(X)$ is norm-dense in $\mathcal{B}(X)$ whenever X satisfies the Radon–Nikodym Property or the so-called property α, and deduce a quite general renorming result, but the question if $\mathcal{B}_a(X)$ is dense in $\mathcal{B}(X)$ for an arbitrary Banach space X arises. Answering this question in the negative is the main purpose of this note. We show that a Banach space used by W. Gowers in the study of norm-attaining operators works as a counterexample. Actually, with a slightly more involved construction, one can get a stronger result. We produce an example of a Banach space X such that $NA(X, X^*)$ is not dense in $L(X, X^*)$ and it clearly follows that $\mathcal{B}_a(X)$ is not dense in $\mathcal{B}(X)$.

We also deal with the "quadratic" version of the Bishop–Phelps Theorem. The argument used in [A-F-W, Theorem 2] can be easily modified to show the density of norm-attaining quadratic forms on a Banach space with the Radon–Nikodym Property. As the reader will probably guess, we show that this density does not hold for an arbitrary Banach space by using the same counterexample as in the bilinear case. This answers in the negative a question posed by P. Georgiev at the conference on Functional Analysis and Applications held in Gargnano (Italy) in 1993.

Our results are valid in the real as well as in the complex case and we try to use a notation which covers both cases. There is an open question concerning the complex version of the general Bishop–Phelps Theorem for nonbalanced sets (see [P] and [D-G-Z, Problem I.4]). Nevertheless, when dealing with norm-attaining functionals or operators, only a balanced set (the unit ball) is involved and our discussion is unrelated to the above-mentioned problem.

Let us start by recalling the definition of the Banach space G used by W. Gowers to show that l_p fails Lindenstrauss' property B for $1 < p < \infty$ [G, Appendix].

Definition 1: For a scalar sequence x and $n \in \mathbb{N}$ let us write

$$\Phi_n(x) = \frac{1}{H_n} \sup\left\{ \sum_{j \in J} |x(j)| : J \subset \mathbb{N},\ |J| = n \right\}$$

where $|J|$ is the cardinality of the set J and $H_n = \sum_{k=1}^{n} k^{-1}$. We will denote by G the Banach space of those sequences x such that

$$\lim_{n \to \infty} \Phi_n(x) = 0$$

with the norm given by

$$\|x\| = \sup\{\Phi_n(x): n \in \mathbb{N}\} \quad (x \in G)$$

and $\{e_n\}$ will be the unit vector basis of G.

In the following lemma we recall the properties of the space G that will be needed below.

LEMMA 2 (Gowers [G], see also [A-A-P]):

(i) *The unit ball of G lacks extreme points. In fact, for every $x \in G$ with $\|x\| = 1$, there exist a natural number m and $\delta > 0$ such that $\|x + \lambda e_k\| = 1$ for $k \geq m$ and any scalar λ with $|\lambda| \leq \delta$.*

(ii) *For $1 < p < \infty$, G is contained in l_p and the formal identity from G into l_p is a bounded operator.*

From property (i) we deduce the following:

PROPOSITION 3:

(i) *If φ is a norm-attaining continuous bilinear form on $G \times G$, then*

$$\varphi(e_m, e_n) = 0$$

for large enough m and n.

(ii) *If a continuous quadratic form Q on G attains its norm at a point $x_0 \in B_G$, then*

$$Q(x_0)Q(e_n) \leq 0$$

for large enough n.

Proof: (i) Let x_0, y_0 be norm-one vectors in G such that

$$|\varphi(x, y)| \leq |\varphi(x_0, y_0)|$$

for all x, y in the unit ball of G, and using Lemma 2(i), let $\delta > 0$ and $N \in \mathbb{N}$ be such that

$$\|x_0 \pm \delta e_n\| = \|y_0 \pm \delta e_n\| = 1 \quad (n \geq N).$$

For $m \geq N$, from

$$|\varphi(x_0 \pm \delta e_m, y_0)| = |\varphi(x_0, y_0) \pm \delta\varphi(e_m, y_0)| \leq |\varphi(x_0, y_0)|$$

and the strict convexity of the scalar field, we get that

$$\varphi(e_m, y_0) = 0.$$

An analogous argument shows that $\varphi(x_0, e_n) = 0$ for $n \geq N$. Then we have

$$|\varphi(x_0 \pm \delta e_m, y_0 \pm \delta e_n)| = |\varphi(x_0, y_0) \pm \delta^2 \varphi(e_m, e_n)| \leq |\varphi(x_0, y_0)|$$

and again the strict convexity of the scalar field gives

$$\varphi(e_m, e_n) = 0 \quad (m, n \geq N)$$

as required.

(ii) Assume without loss of generality that $\|Q\| = Q(x_0) = 1$ and let ϕ be the continuous symmetric sesquilinear form on $G \times G$ such that

$$Q(x) = \phi(x, x), \quad \forall x \in G.$$

As in the first part of the proof, let $\delta > 0$ and $N \in \mathbb{N}$ be such that

$$\|x_0 + \lambda e_n\| \leq 1,$$

for $n \geq N$ and any scalar λ with $|\lambda| \leq \delta$. By taking $\lambda = \rho\mu$ with $|\mu| = 1$ and $0 \leq \rho \leq \delta$, we have the inequality

$$1 = Q(x_0) \geq |Q(x_0 + \rho\mu e_n)| = |1 + 2\rho \operatorname{Re} \mu\phi(x_0, e_n) + \rho^2 Q(e_n)|,$$

hence

$$2\rho \operatorname{Re} \mu\phi(x_0, e_n) + \rho^2 Q(e_n) \leq 0. \tag{1}$$

Dividing by ρ and letting $\rho \to 0$ gives

$$\operatorname{Re} \mu\phi(x_0, e_n) \leq 0,$$

but this is true for any scalar μ with $|\mu| = 1$, so $\phi(x_0, e_n) = 0$. It now follows from (1) that $Q(e_n) \leq 0$ for $n \geq N$, as required. ∎

COROLLARY 4: *The set $\mathcal{B}_a(G)$ of norm-attaining continuous bilinear forms on $G \times G$ is not dense in the space $\mathcal{B}(G)$ of all continuous bilinear forms.*

Proof: In view of Lemma 2(ii) we can define a continuous bilinear form on $G \times G$ by

$$\psi(x,y) = \sum_{n=1}^{\infty} x(n)y(n) \quad (x,y \in G).$$

Since $\psi(e_n, e_n) = 1$ for all n, by taking a large enough n and using the above proposition we have

$$\|\psi - \varphi\| \geq 1$$

for arbitrary $\varphi \in \mathcal{B}_a(G)$, so ψ cannot be approximated by norm-attaining bilinear forms. ∎

COROLLARY 5: *There is a continuous quadratic form on G which cannot be approximated by norm-attaining quadratic forms.*

Proof: Define

$$Q_0(x) = \sum_{n=1}^{\infty} |x(n)|^2 \quad (x \in G)$$

to get a continuous quadratic form on G. By using the fact that Q_0 is positive definite, one can easily show that if a quadratic form Q attains its norm at $x_0 \in B_G$ and Q is close enough to Q_0, then $Q(x_0) \geq 0$. By the second part of Proposition 3, $Q(e_n) \leq 0$ for n large enough, so $\|Q - Q_0\| \geq 1$. ∎

The question of the density of norm-attaining bilinear forms admits a slightly different approach that will allow an improvement of Corollary 4. Recall that for any Banach space X, there is a natural identification between $\mathcal{B}(X)$ and the space $L(X, X^*)$. Under this identification, the bilinear form $\varphi \in \mathcal{B}(X)$ becomes the operator $T \in L(X, X^*)$ given by

$$\langle T(x), y\rangle = \varphi(x,y) \quad (x,y \in X).$$

It is clear that T attains its norm whenever φ does, but simple examples show that the converse is not true. Thus, we can consider that $\mathcal{B}_a(X)$ is contained in the set $NA(X, X^*)$ of norm-attaining operators. Our next result gives a procedure to get examples of Banach spaces X such that $NA(X, X^*)$ is not dense in $L(X, X^*)$ even less can $\mathcal{B}_a(X)$ be dense in $\mathcal{B}(X)$.

THEOREM 6: *Let Y be a Banach space such that Y^* is strictly convex and there is a noncompact operator from G into Y^*. Consider the Banach space $X = G \oplus_1 Y$ (direct sum in the l_1-sense). Then $NA(X, X^*)$ is not dense in $L(X, X^*)$.*

Proof: Let $A \in L(G, Y^*)$ be noncompact and define an operator T from X into $X^* \cong G^* \oplus_\infty Y^*$ by

$$T(z, y) = (0, A(z)) \quad (z \in G, y \in Y).$$

By [A-A-P, Theorem 1.4], A cannot be approximated by norm-attaining operators and a routine argument will show that the same happens with T. Suppose, on the contrary, that for any $\varepsilon > 0$ there is $S \in NA(X, X^*)$ such that $\|S - T\| < \varepsilon$. Let P and Q be the natural projections from X onto G and from X^* onto Y^*, respectively. Since $QT = T$ we will have $\|(\mathrm{Id} - Q)S\| < \|QS\| = \|S\|$, provided that ε is small enough. It follows easily that QS attains its norm, but $\|QS - T\| < \varepsilon$, so we can suppose $S = QS$. On the other hand, we also have $T(\mathrm{Id} - P) = 0$, so

$$\|S(\mathrm{Id} - P)\| < \|S\|, \tag{1}$$

again for small enough $\varepsilon > 0$.

For $x \in X$ with $Px \neq x$, it follows that

$$\begin{aligned} \|Sx\| &\leq \|SP\|\|Px\| + \|S(\mathrm{Id} - P)\|\|x - Px\| \\ &< \|S\|(\|Px\| + \|x - Px\|) = \|S\|\|x\|. \end{aligned}$$

Therefore, if S attains its norm at a point $x_0 \in S_X$, then $Px_0 = x_0$ and SP also attains its norm at x_0.

Note finally that

$$\|SP - A\| = \|QSP - QTP\| < \varepsilon,$$

hence A can be approximated by norm-attaining operators, a contradiction. ∎

Examples 7: (a) By Lemma 2(ii) there is a noncompact operator from G into the strictly convex space $l_q = l_p^*$ $(1 < p < \infty)$. By the above theorem the space $X = G \oplus_1 l_p$ is such that $NA(X, X^*)$ is not dense in $L(X, X^*)$.

(b) Actually, one can get the Banach space X in the previous theorem to be isomorphic to G. Since G is separable, there exists a Banach space Y isomorphic to G such that Y^* is strictly convex (see [D-G-Z, Theorem II.2.6] for example). Moreover, there is a noncompact operator from G into G^*, so Y satisfies the requirements in the above theorem, but this time $X = G \oplus_1 Y$ is isomorphic to G. We do not know if $NA(G, G^*)$ is dense in $L(G, G^*)$ for the original norm on G.

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